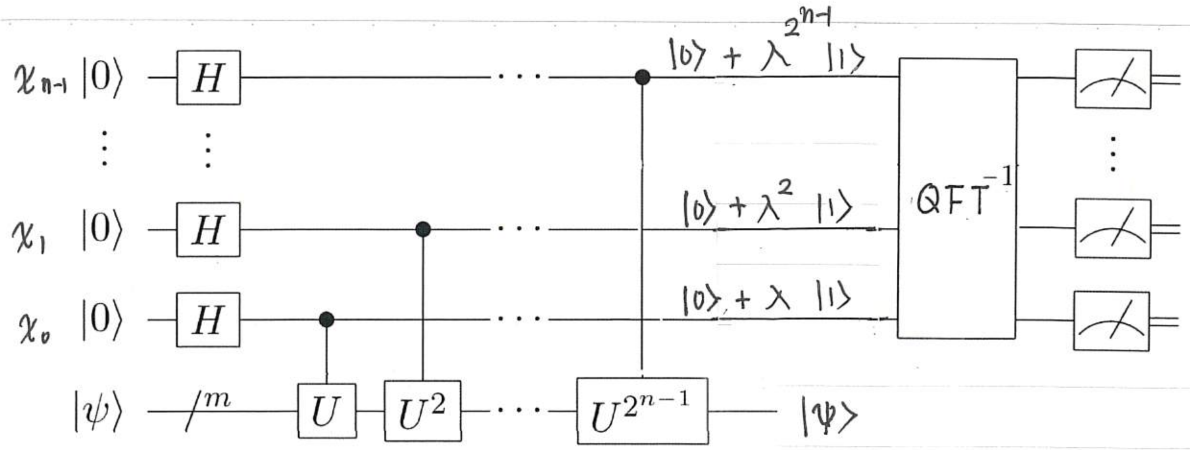
Phase estimation algorithm



Input: $\begin{cases} U : \text{unitary} \\ |\psi\rangle : \text{eigen-vector} \end{cases}$ $\quad U|\psi\rangle = \underset{\text{eigen value}}{\lambda} |\psi\rangle, \quad |\lambda| = 1, \quad \lambda = e^{2\pi i \theta}, \quad 0 \le \theta < 1$

$$\begin{cases} \theta = 0.\underline{\theta_1\theta_2\cdots\theta_n}\,\theta_{n+1}\cdots \\ 2^n\theta = \underline{\theta_1\theta_2\cdots\theta_n}.\theta_{n+1}\cdots \end{cases}$$

Output: $\lfloor 2^n\theta \rfloor = \theta_1\theta_2\cdots\theta_n$

$$\frac{1}{\sqrt{2}}\left(|0\rangle + \lambda^{2^{n-1}}|1\rangle\right) \otimes \cdots \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + \lambda^2|1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + \lambda|1\rangle\right) \otimes |\psi\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x_{n-1}=0}^{1} \lambda^{x_{n-1}2^{n-1}} |x_{n-1}\rangle \otimes \cdots \otimes \sum_{x_1=0}^{1} \lambda^{x_1 2} |x_1\rangle \otimes \sum_{x_0=0}^{1} \lambda^{x_0} |x_0\rangle \otimes |\psi\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{(x_{n-1}\cdots x_0)\in \mathbb{B}^n} \lambda^{x_{n-1}2^{n-1} + \cdots + x_1 2 + x_0} |x_{n-1}\cdots x_1 x_0\rangle \otimes |\psi\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} \lambda^x |x\rangle \cdots$$

$$= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} e^{\frac{2\pi i}{2^n} 2^n \theta x} |x\rangle \cdots \quad \begin{pmatrix} N = 2^n \\ \omega = e^{\frac{2\pi i}{2^n}} \end{pmatrix}$$

$$= \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} \omega^{N\theta x} |x\rangle \quad \left(\xrightarrow[N\theta\in\mathbb{Z}]{QFT^{-1}} |N\theta\rangle \right)$$

$$\xrightarrow{QFT_N^{-1}} \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} \omega^{N\theta x} \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{-kx} |k\rangle$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \underbrace{\sum_{x=0}^{N-1} \omega^{(N\theta-k)x}}_{p_k} |k\rangle$$

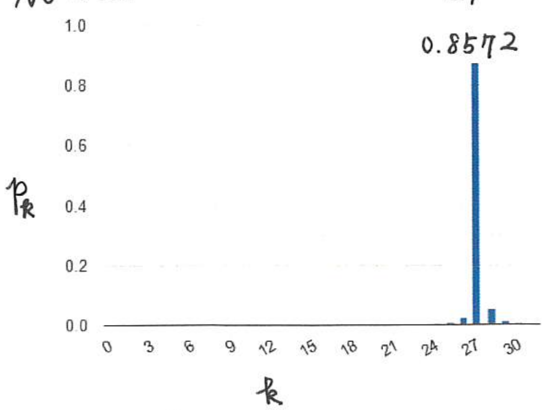
$$= \frac{1}{N}\left(p_0|0\rangle + p_1|1\rangle + \cdots + p_{N-1}|N-1\rangle\right)$$

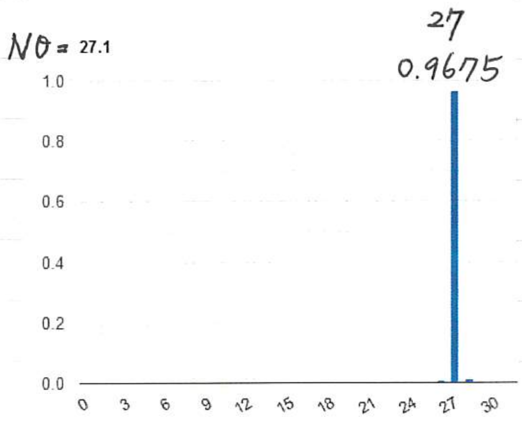
$N = 32$

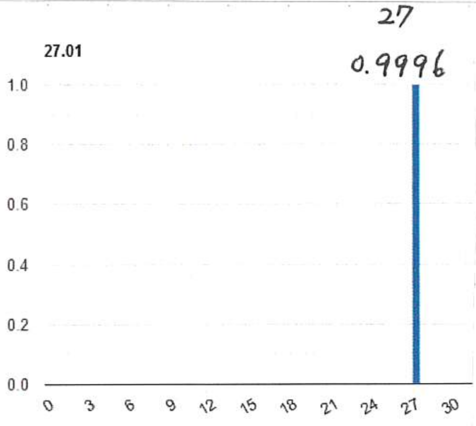
$N\theta =$ 27.2

27

0.8572







$n\theta$	p_{27}
27.4	0.573082
27.3	0.737053
27.2	0.875253
27.1	0.967562
27.09	0.973660
27.08	0.979141
27.07	0.983999
27.06	0.988224
27.05	0.991810
27.04	0.994752
27.03	0.997046
27.02	0.998686
27.01	0.999671

$$p_k = \frac{1}{N}\sum_{x=0}^{N-1}\omega^{(N\theta-k)x} = \begin{cases} 1 & k = N\theta \\ \frac{1}{N}\,\frac{1-\omega^{N(N\theta-k)}}{1-\omega^{N\theta-k}} & k \neq N\theta \end{cases}$$

$k \neq N\theta$

$$Pr(|k\rangle) = |p_k|^2 = \frac{1}{N^2}\left|\frac{1-\omega^{N(N\theta-k)}}{1-\omega^{N\theta-k}}\right|^2$$

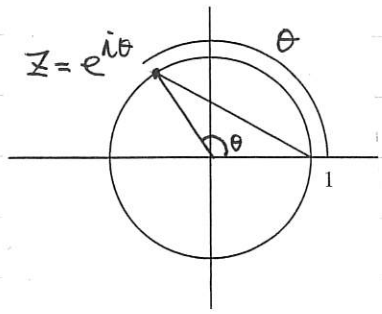
$$= \frac{1}{N^2}\left|\frac{\sin\frac{\pi}{N}N(N\theta-k)}{\sin\frac{\pi}{N}(N\theta-k)}\right|^2$$

取 $k=d$, $|N\theta-d| = |\varepsilon| \le \frac{1}{2}$ (最靠近 $N\theta$ 的整數)

$$Pr(|d\rangle) = \frac{1}{N^2}\left|\frac{1-e^{\frac{2\pi i}{N}N|\varepsilon|}}{1-e^{\frac{2\pi i}{N}|\varepsilon|}}\right|^2$$

$$\ge \frac{1}{N^2}\left|\frac{\frac{2}{\pi}2\pi|\varepsilon|}{\frac{2\pi}{N}|\varepsilon|}\right|^2 = \frac{4}{\pi^2} > 0.4$$

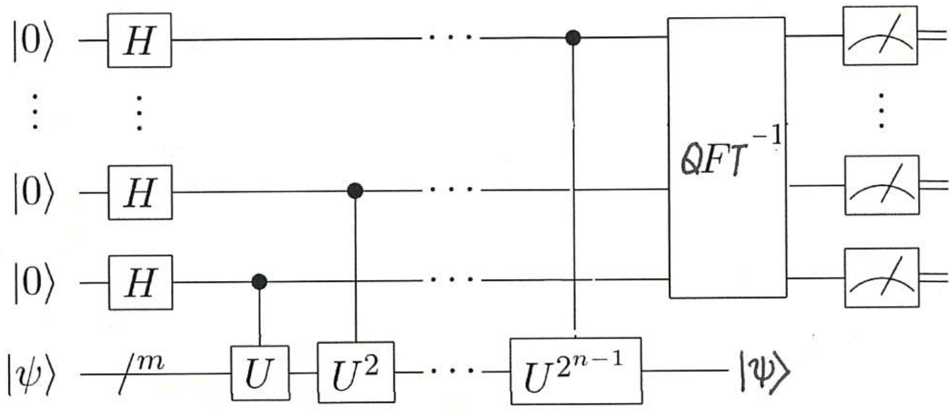
$(2\pi|\varepsilon| \le \pi)$



$$1 < \frac{\theta}{|1-z|} = \frac{\theta}{2\sin\frac{\theta}{2}} \le \frac{\pi}{2}, \quad 0<\theta\le\pi$$

$$\therefore z = e^{i\theta}, \quad \frac{2\theta}{\pi} < |1-z| < \theta$$

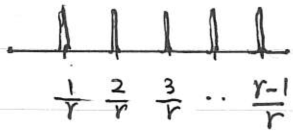
$$\left(x\in\left(0,\tfrac{\pi}{2}\right),\ f(x) = \frac{x}{\sin x} \uparrow \frac{\pi}{2}, \ \because f'(x) > 0 \right)$$



(A) $|\psi\rangle$ $\qquad U|\psi\rangle = e^{2\pi i\theta}|\psi\rangle$ $\qquad |N\theta\rangle \xrightarrow[N]{} \theta$

(B) $|\psi_d\rangle$ $\qquad U|\psi_d\rangle = e^{\frac{2\pi i}{r}d}|\psi_d\rangle$ $\qquad |N\frac{d}{r}\rangle \xrightarrow[N]{} \frac{d}{r}$

(C) $|1\rangle$ $\qquad \frac{1}{\sqrt{r}}\sum_{d=0}^{r-1}|N\frac{d}{r}\rangle \xrightarrow[N]{} \frac{d}{r}$ (some d)



定理

$$\begin{cases} U|y\rangle = |ay \bmod M\rangle \\ |\psi_d\rangle = \frac{1}{\sqrt{r}}\left(|1\rangle + \frac{1}{\omega^d}|a\rangle + \frac{1}{\omega^{2d}}|a^2\rangle + \cdots + \frac{1}{\omega^{(r-1)d}}|a^{r-1}\rangle\right) \end{cases}$$

則
$$\begin{cases} \text{(i) } U|\psi_d\rangle = \omega^d|\psi_d\rangle \\ \text{(ii) } \frac{1}{\sqrt{r}}\sum_{d=0}^{r-1}|\psi_d\rangle = |1\rangle \end{cases}$$

$$\begin{cases} a^r \equiv 1 \mod M \\ \omega = e^{\frac{2\pi i}{r}}, \quad \omega^r = 1 \end{cases}$$

$(r=5)$

證明 (i) $U|\psi_3\rangle = \frac{1}{\sqrt{5}}\left(|a\rangle + \frac{1}{\omega^3}|a^2\rangle + \frac{1}{\omega^6}|a^3\rangle + \frac{1}{\omega^9}|a^4\rangle + \frac{1}{\omega^{12}}|a^5\rangle\right)$

$\omega^3|\psi_3\rangle = \frac{1}{\sqrt{5}}\left(\omega^3|1\rangle + |a\rangle + \frac{1}{\omega^3}|a^2\rangle + \frac{1}{\omega^6}|a^3\rangle + \frac{1}{\omega^9}|a^4\rangle\right)$

(ii)
$$\frac{1}{\sqrt{5}}\begin{cases} |\psi_0\rangle = \frac{1}{\sqrt{5}}\left(|1\rangle + |a\rangle + |a^2\rangle + |a^3\rangle + |a^4\rangle\right) \\ |\psi_1\rangle = \frac{1}{\sqrt{5}}\left(|1\rangle + \frac{1}{\omega}|a\rangle + \frac{1}{\omega^2}|a^2\rangle + \frac{1}{\omega^3}|a^3\rangle + \frac{1}{\omega^4}|a^4\rangle\right) \\ |\psi_2\rangle = \frac{1}{\sqrt{5}}\left(|1\rangle + \frac{1}{\omega^2}|a\rangle + \frac{1}{\omega^4}|a^2\rangle + \frac{1}{\omega^6}|a^3\rangle + \frac{1}{\omega^8}|a^4\rangle\right) \\ |\psi_3\rangle = \frac{1}{\sqrt{5}}\left(|1\rangle + \frac{1}{\omega^3}|a\rangle + \frac{1}{\omega^6}|a^2\rangle + \frac{1}{\omega^9}|a^3\rangle + \frac{1}{\omega^{12}}|a^4\rangle\right) \\ +)\ |\psi_4\rangle = \frac{1}{\sqrt{5}}\left(|1\rangle + \frac{1}{\omega^4}|a\rangle + \frac{1}{\omega^{4\cdot 2}}|a^2\rangle + \frac{1}{\omega^{4\cdot 3}}|a^3\rangle + \frac{1}{\omega^{4\cdot 4}}|a^4\rangle\right) \end{cases}$$

$|1\rangle$